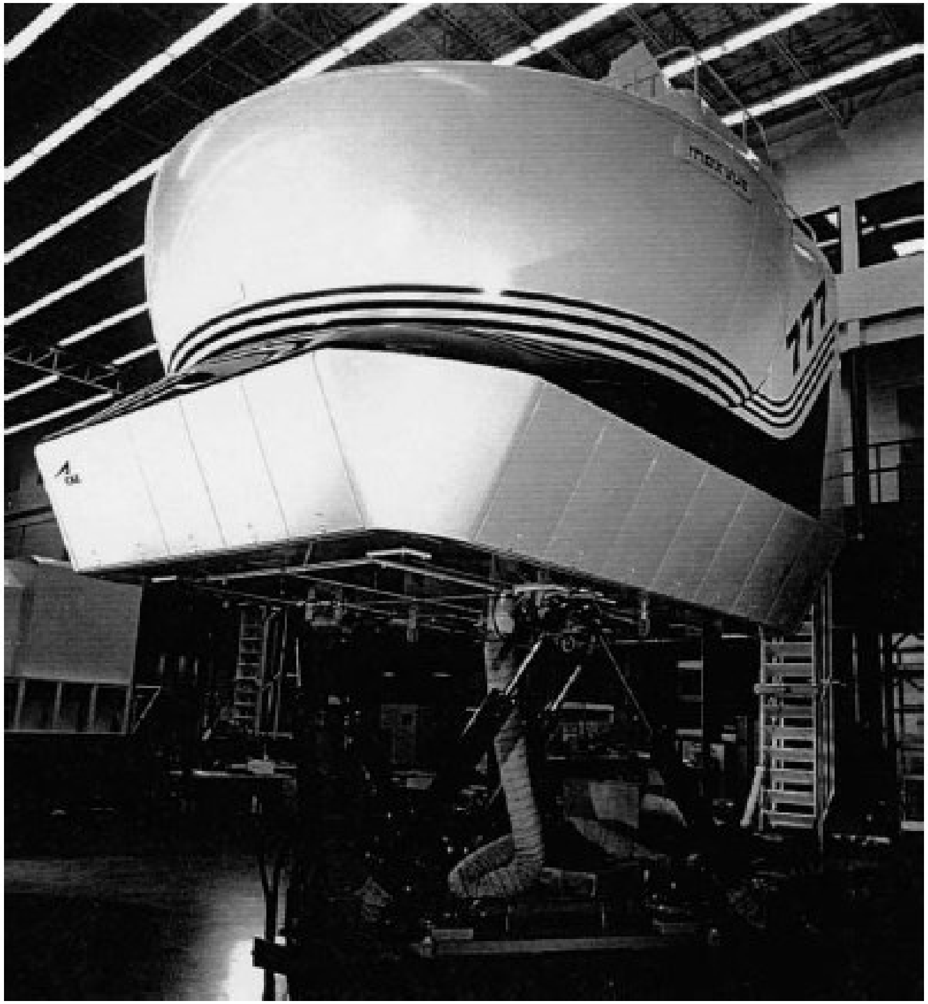
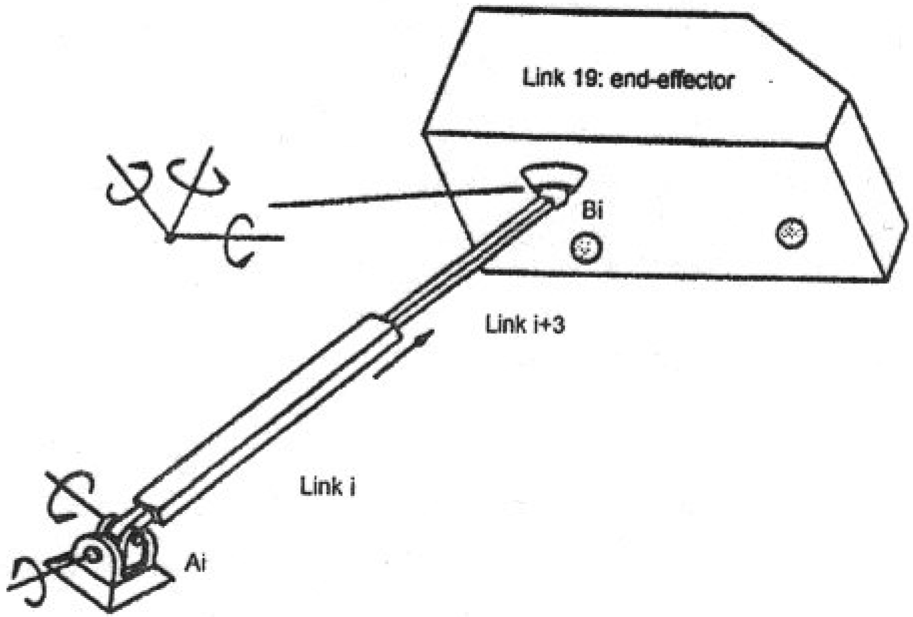
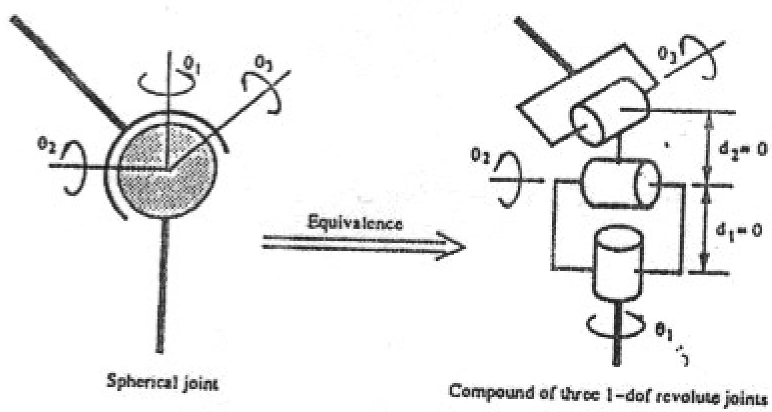
DYNAMICS OF PARALLEL MANIPULATOR

PARALLEL MANIPULATORS



6-degree of Freedom Flight Simulator


Link 19: end-effector
Bi
Link i+3
Link i
Ai


θ1
θ3
θ2
Equivalence
d2= 0
d1= 0
Spherical joint
Compound of three 1-dof revolute joints

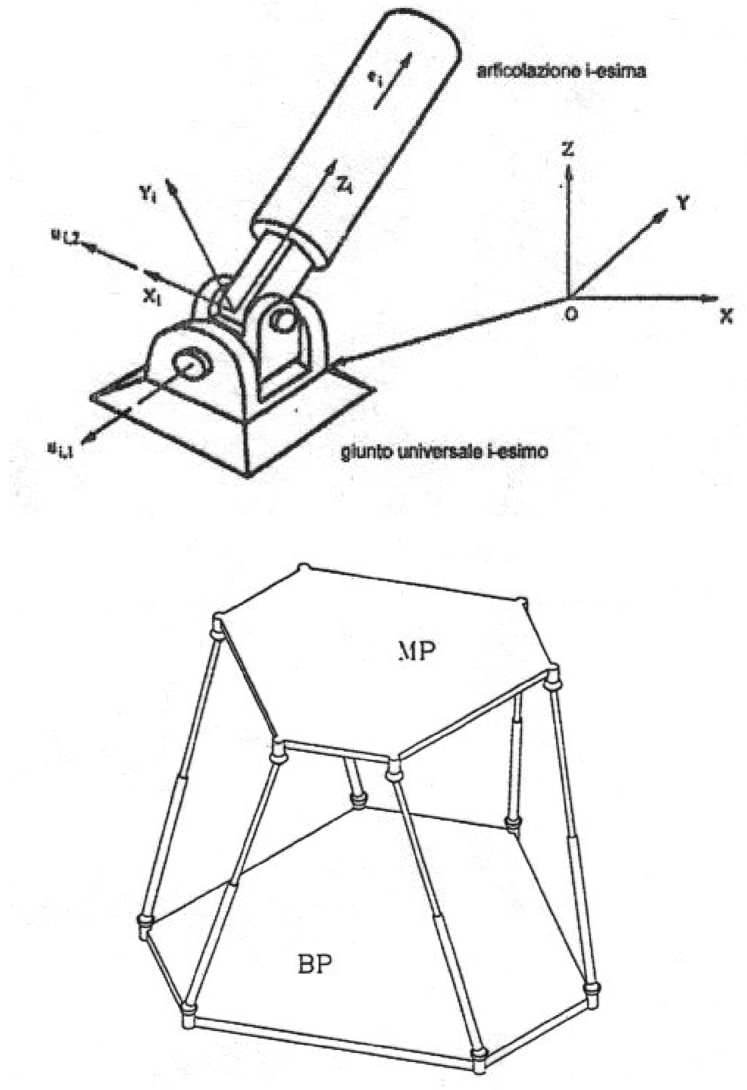
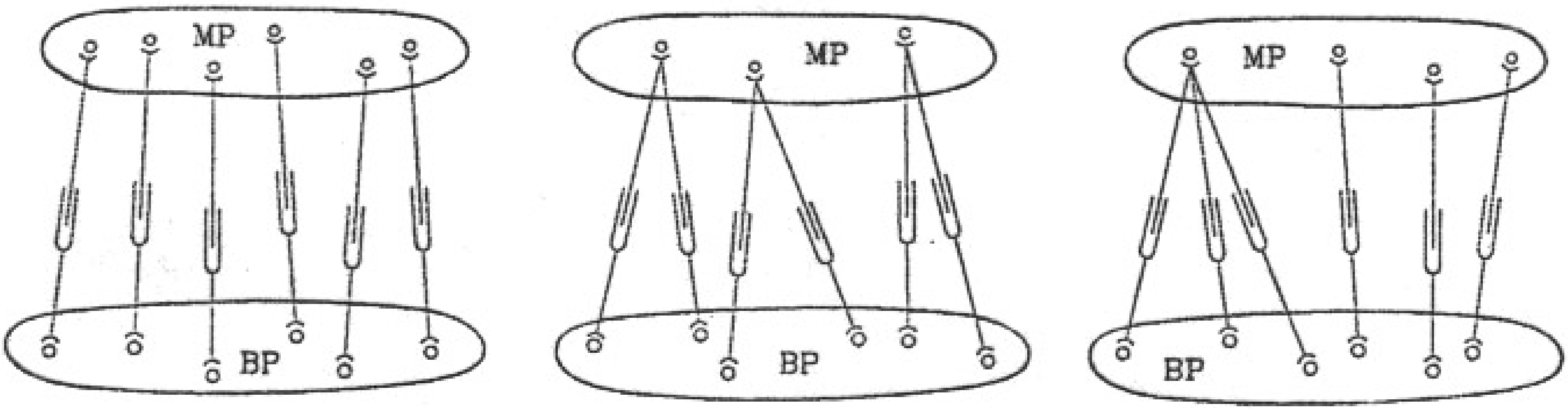


Figure 1: The six-DOF parallel manipulator with prismatic legs

BACKGROUND

Platform-type parallel mechanisms

6-DOF MANIPULATORS



(a) The generalized 6-6 structure with single spherical joints

(b) A class of 6-3 structure with double spherical joints

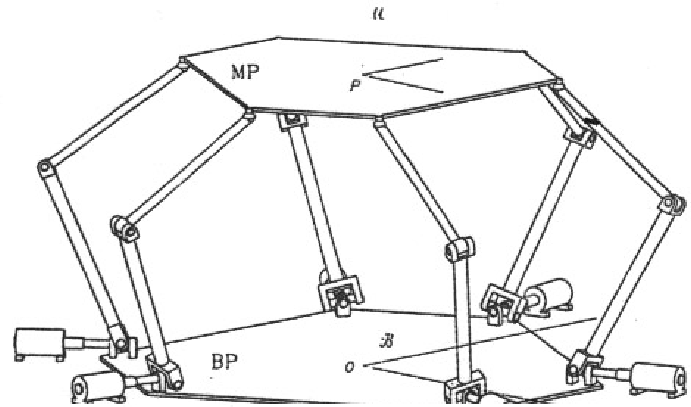
(c) A class of 6-4 structure with a triple spherical joint

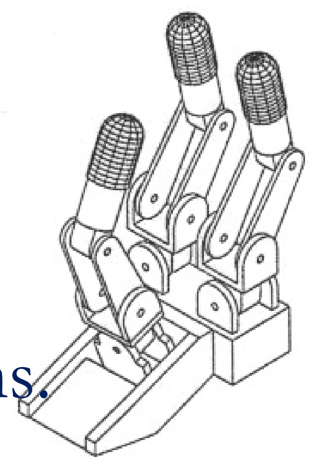
INTRODUCTION

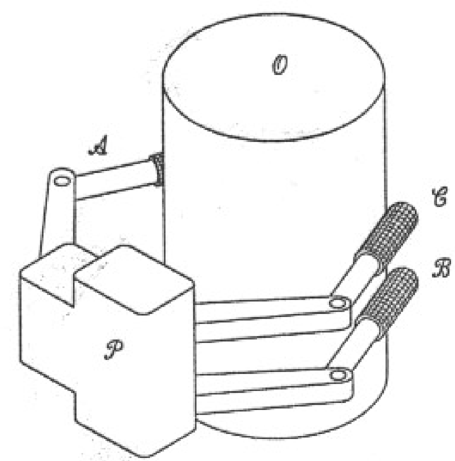
Under alternative robotic mechanical systems we understand here:

- Parallel Robots;
- Multifingered hands;
- Walking Machines;
- Rolling Robots.

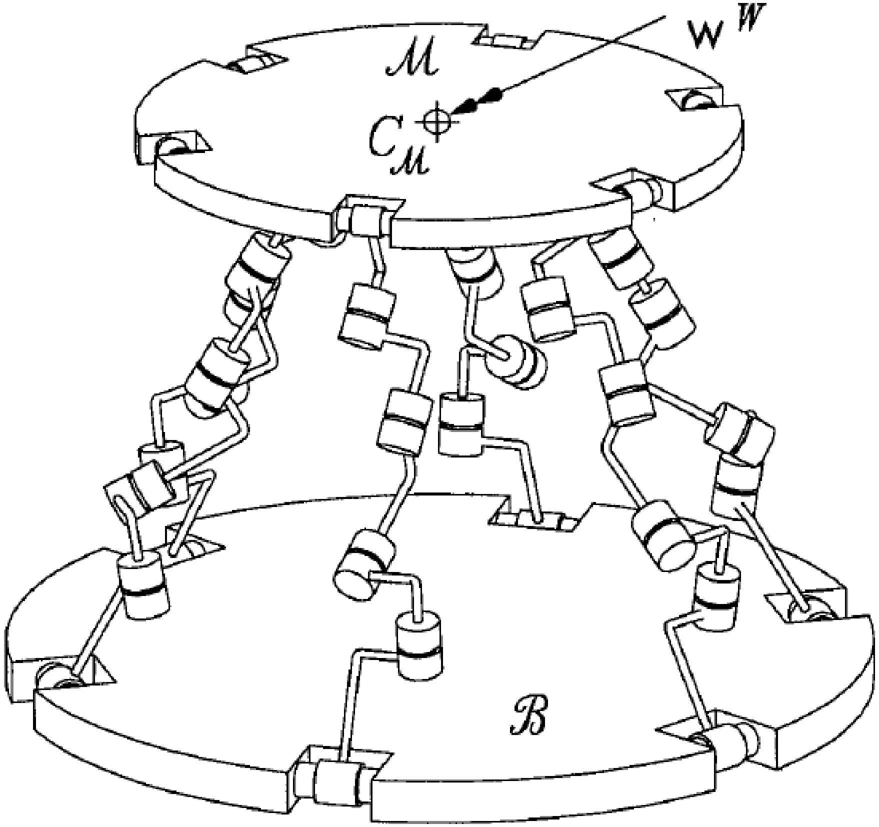
Parallel manipulators are composed of kinematic chains with closed subchains.







KINEMATICS OF PARALLEL MANIPULATORS



General six-dof parallel manipulator

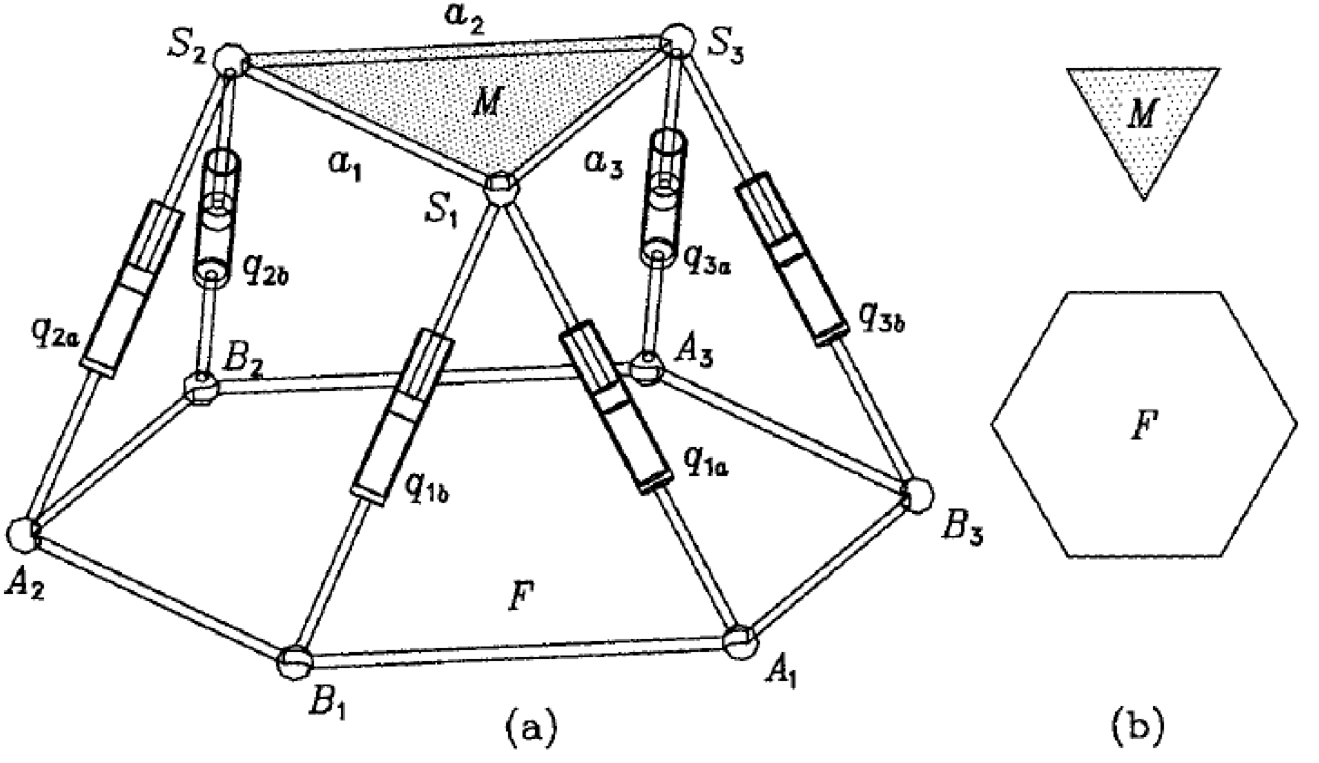
KINEMATICS OF PARALLEL MANIPULATORS

We can distinguish two platforms:

- One fixed to the ground $\mathcal{B}$
- One capable of moving arbitrarily within its workspace $\mathcal{M}$.

The moving platform is connected to the fixed platform through six legs, each being regarded as a six-axis serial manipulator whose base is $\mathcal{B}$
and whose end-effector is $\mathcal{M}$.
The whole leg is composed of six links coupled through six revolutes.

KINEMATICS OF PARALLEL MANIPULATORS



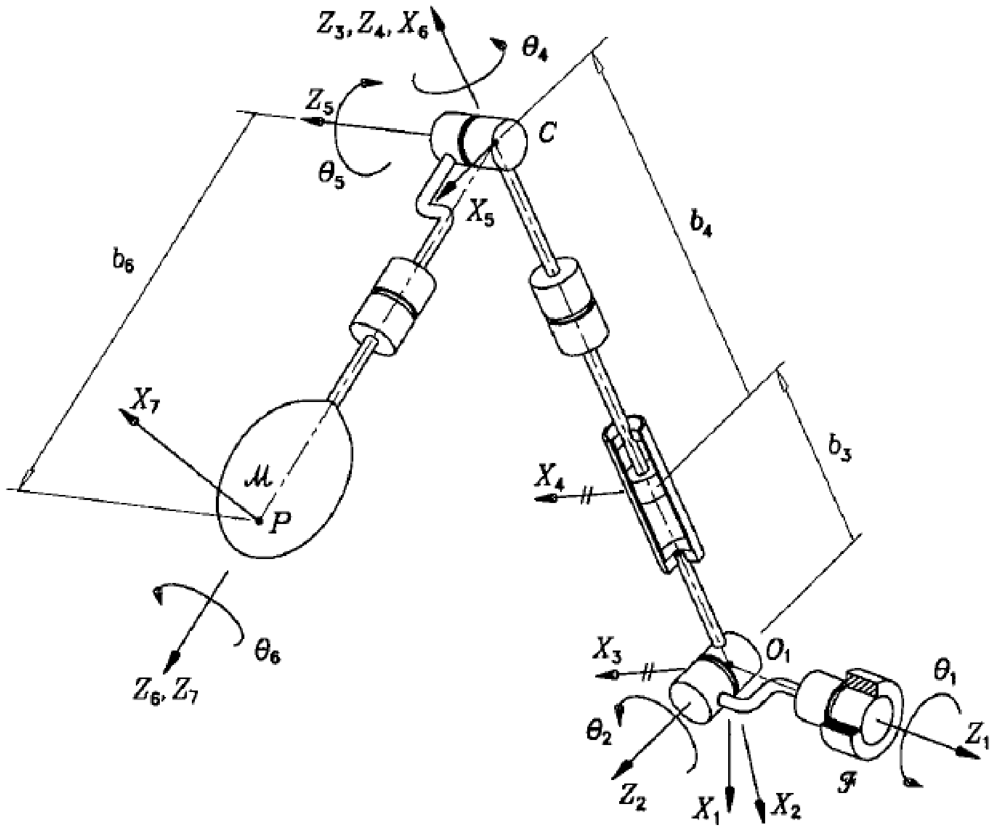
Six-dof flight simulator: (a) general layout; (b) geometry of its two platforms

KINEMATICS OF PARALLEL MANIPULATORS

In this figure, the fixed platform $\mathcal{B}$ is a regular hexagon, while the moving platform $\mathcal{M}$ is an equilateral triangle. Moreover, $\mathcal{B}$ is connected to $\mathcal{M}$ by means of six serial chains, each comprising five revolutes and one prismatic pair. Three of the revolutes bear concurrent axes, and hence, constitute a spherical joint, while two more have axes intersecting at right angles, thus constituting a universal joint.

It is to be noted that although each leg of the manipulator has a spherical joint at only one end and a universal joint at the other end.

KINEMATICS OF PARALLEL MANIPULATORS



A layout of a leg of the manipulator

KINEMATICS OF PARALLEL MANIPULATORS

We analyze the inverse kinematics of one leg of the manipulator. The Denavit-Hartenberg parameters of the leg shown in this figure are given in table:

i	ai	bi	αi
1	0	0	90°
2	0	0	90°
3	0	b_3	0°
4	0	b_4 (const)	90°
5	0	0	90°
6	0	b_6(const)	0°

KINEMATICS OF PARALLEL MANIPULATORS

It is apparent that the leg under study is a decoupled manipulator. In view of the DH parameters of this manipulator, we have:

$$\mathbf{Q}_1\mathbf{Q}_2(\mathbf{a}_3 + \mathbf{a}_4) = \mathbf{c}$$

where $\mathbf{c}$ denotes the position vector of the center C of the spherical wrist and, since frames $\mathcal{F}_3$ and $\mathcal{F}_4$ of the DH notation are related by a pure translation, $\mathbf{Q}_3 = \mathbf{1}$.

Upon equating the squares of the Euclidean norms of both sides of the foregoing equation, we obtain:

$$\|\mathbf{a}_3 + \mathbf{a}_4\|^2 = \|\mathbf{c}\|^2$$

KINEMATICS OF PARALLEL MANIPULATORS

Where, by virtue of the DH parameters of Table:

$$\|\mathbf{a}_3 + \mathbf{a}_4\|^2 = (b_3 + b_4)^2$$

Now, since both b_3 and b_4 are positive by construction, we have:

$$b_3 = \|\mathbf{c}\| - b_4 > 0$$

Note that the remaining five joint variables of the leg under study are not needed for purposes of inverse kinematics, and hence, their calculation could be skipped.

KINEMATICS OF PARALLEL MANIPULATORS

We derive three scalar equations in two unknowns, θ_1 and θ_2, namely,

$$(b_3 + b_4) = x_c c_1 + y_c s_1$$
$$-(b_3 + b_4 = z_c$$
$$0 = x_c s_1 - y_c c_1$$

in which c_i and s_i stand for $cos\theta i$ and sin θi, respectively

θ_1 is derived as: $\theta_1 = \tan\left(\frac{y_c}{x_c}\right)$

which yields a unique value of θ_1 rather than the two lying π radians apart, for the two coordinates x_C and y_C determine the quadrant in which θ_1 lies.

KINEMATICS OF PARALLEL MANIPULATORS

Once θ_1 is known, θ_2 is derived uniquely from the remaining two equations through its cosine and sine functions, i.e.,

$$c_2 = \frac{z_c}{b_3 + b_4} \qquad s_2 = \frac{x_c c_1 + y_c s_1}{b_3 + b_4}$$

With the first three joint variables of this leg known, the remaining ones, are calculated as described in foregoing chapter.

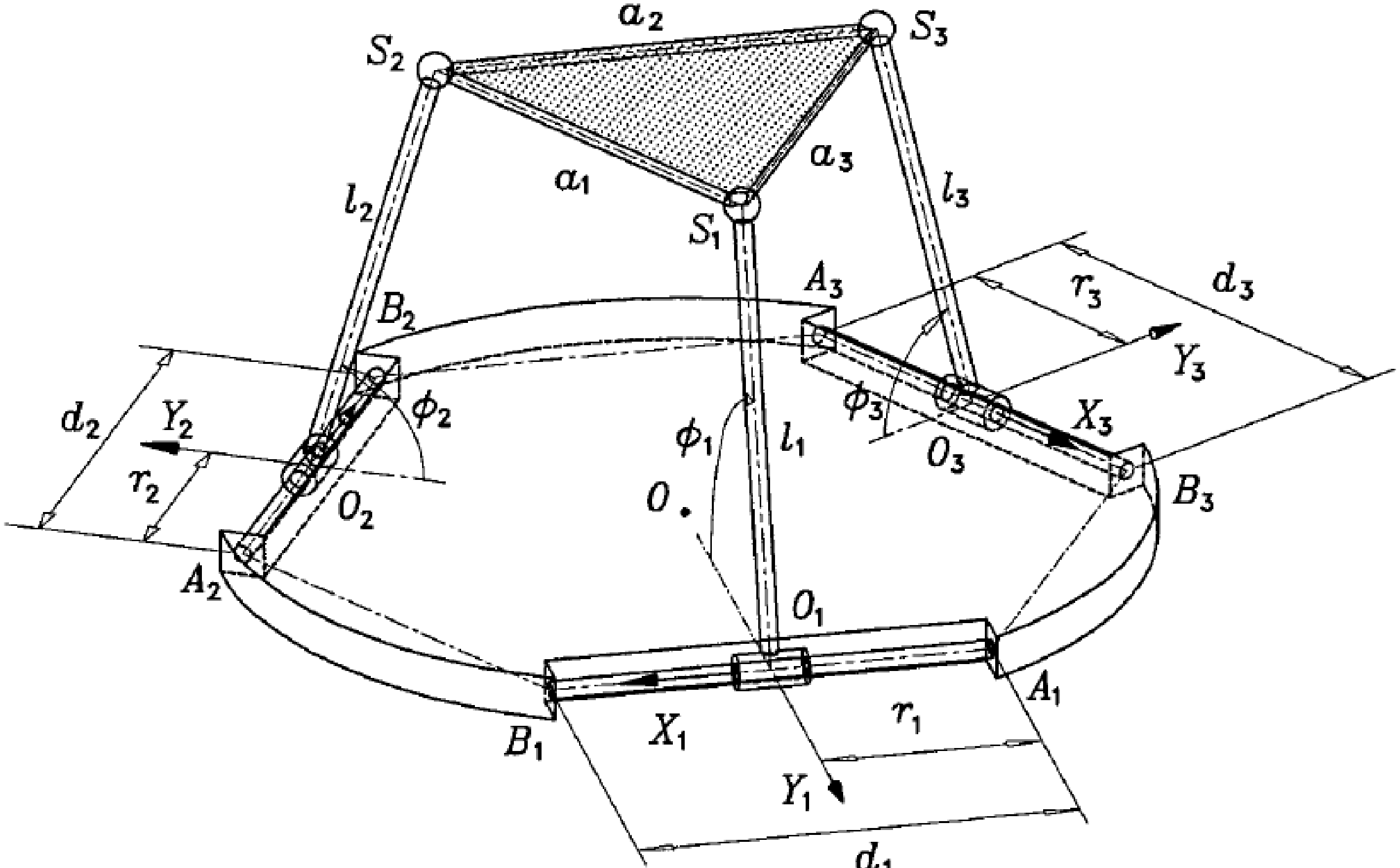
Therefore, the inverse kinematics of each leg admits two solutions, one for the first three variables and two for the last three.

KINEMATICS OF PARALLEL MANIPULATORS

Moreover, since the only actuated joint is one of the first three, which of the two wrist solutions is chosen does not affect the value of b_3, and hence, each manipulator leg admits only one inverse kinematics solution.

While the inverse kinematics of this leg is quite straightforward, its direct kinematics is not. Below we give an outline of the solution procedure for the manipulator under study

KINEMATICS OF PARALLEL MANIPULATORS



Equivalent simplified mechanism

KINEMATICS OF PARALLEL MANIPULATORS

We can replace the pair of legs by a single leg of length l_i connected to the base plate $\mathcal{B}$ by a revolute joint with its axis along A_iB_i. The resulting simplified structure is kinematically equivalent to the original structure.

Now we introduce the coordinate frame $\mathcal{F}_i$ with origin at the attachment point O_i of the ith leg with the base plate $\mathcal{B}$ and the notation below:

For i=1,2,3

- X_i is directed from A_i to B_i
- Y_i is chosen such that Z_i is perpendicular to the plane of the hexagonal base and points upwards.
- O_i is set at the intersection of X_i and Y_i and is designated the center of the revolute joint;

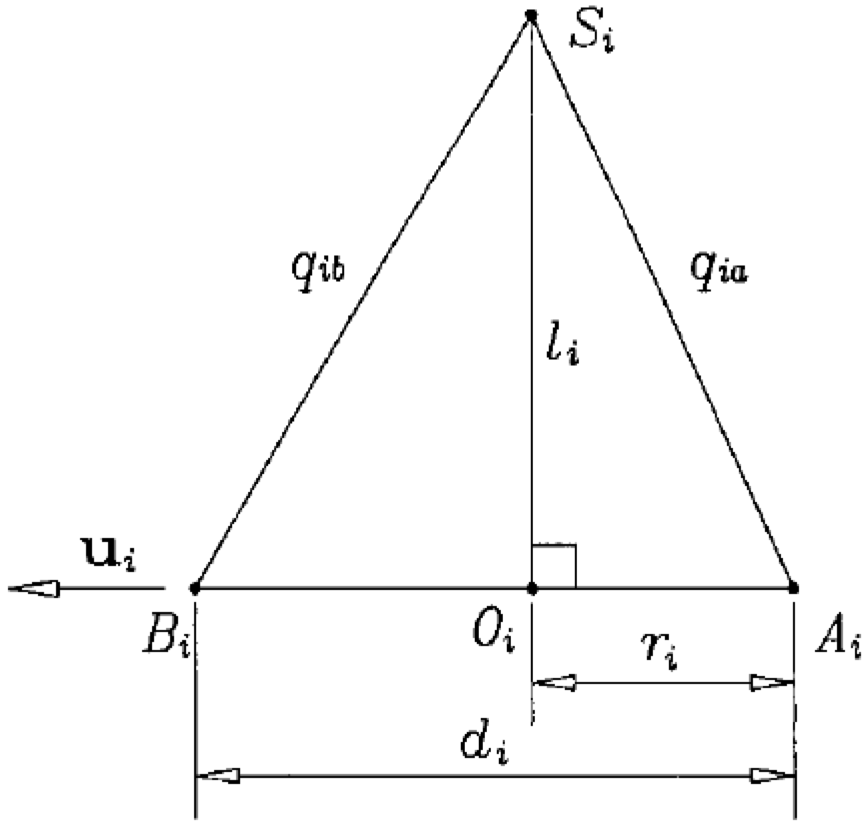
KINEMATICS OF PARALLEL MANIPULATORS

Next, we locate the three vertices S_1, S_2 and S_3 of the triangular plate with position vectors stemming from the center O of the hexagon. Furthermore, we need to determine l_i and O_i.

Referring to Figures and letting $\mathbf{a}_i$ and $\mathbf{b}_i$ denote the position vectors of points A_i and B_i respectively, we have:

$$d_i = \|\mathbf{b}_i - \mathbf{a}_i\|$$

KINEMATICS OF PARALLEL MANIPULATORS



$$r_i = \frac{d_i^2 + q_{ia}^2 - q_{ib}^2}{2d_i}$$

$$l_i = \sqrt{q_{ia}^2 - r_i^2}$$

$$\mathbf{u}_\mathrm{i} = \frac{\mathbf{b}_\mathrm{i} - \mathbf{a}_\mathrm{i}}{\mathrm{d}_\mathrm{i}}$$

for i = 1,2,3, and hence $\mathbf{u}_i$ is the unit vector directed from A_i to B_i.

KINEMATICS OF PARALLEL MANIPULATORS

Moreover, the position of the origin O_i is given by vector $\mathbf{o}_i$, as indicated below:

$$\mathbf{o}_i = \mathbf{a}_i + r_i \mathbf{u}_i \qquad \text{For i= 1,2,3}$$

Furthermore, let $\mathbf{s}_i$ be the position vector of S_i in frame $\mathcal{F}_i$ (O_i, X_i, Y_i, Z_i), then:

$$\mathbf{s}_i = \begin{bmatrix} 0 \\ -l_i cos\phi_i \\ l_i sin\phi_i \end{bmatrix} \qquad \text{For i= 1,2,3}$$

KINEMATICS OF PARALLEL MANIPULATORS

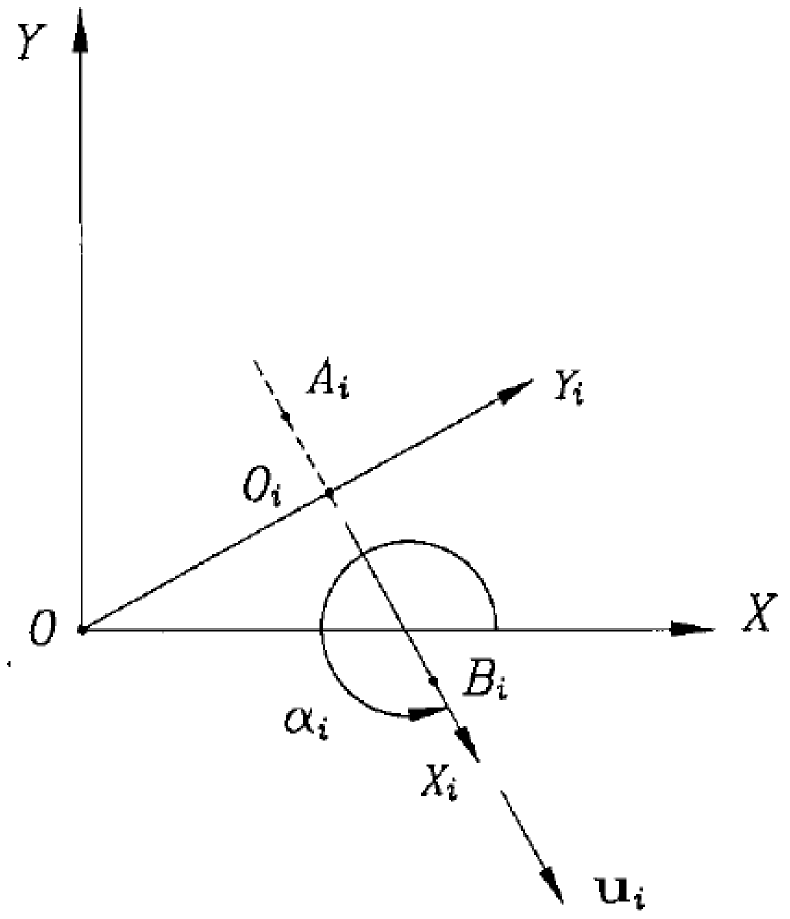
Now a frame $\mathcal{F}_0$ (O, X, Y, Z) is defined with origin at O and axes X and Y in the plane of the base hexagon, and related to X_i and Y_i as depicted in figure. When expressed in frame $\mathcal{F}_0$, $\mathbf{s}_i$ takes on the form:

$$[\mathbf{s}_i]_0 = [\mathbf{o}_i]_0 + [\mathbf{R}_i]_0\mathbf{s}_i \qquad \text{For } i=1,2,3$$

Where $[\mathbf{R}_i]_0$ is the matrix that rotate $\mathcal{F}_0$ to frame $\mathcal{F}_i$, expressed in and given as:

$$[\mathbf{R}_i]_0 = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & 0 \\ \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

KINEMATICS OF PARALLEL MANIPULATORS



Referring to Figure:

$$\cos\alpha_i = \mathbf{u}_\mathrm{i} \cdot \mathbf{i} = u_{\mathrm{ix}}$$

$$\sin\alpha_i = \mathbf{u}_\mathrm{i} \cdot \mathbf{j} = u_{\mathrm{iy}}$$

KINEMATICS OF PARALLEL MANIPULATORS

After substitution we obtain:

$$[\mathbf{s}_{\mathrm{i}}]_0 = [\mathbf{o}_{\mathrm{i}}]_0 + \mathrm{l}_{\mathrm{i}} \begin{bmatrix} u_{ix}\cos\phi_i \\ -u_{iy}\cos\phi_i \\ \sin\phi_i \end{bmatrix}$$ For i=1,2,3

Since the distances between the three vertices of the triangular plate are fixed, the position vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ must satisfy the constraints below:

$$\|\mathbf{s}_2 - \mathbf{s}_1\|^2 = a_1^2$$
$$\|\mathbf{s}_3 - \mathbf{s}_2\|^2 = a_2^2$$
$$\|\mathbf{s}_1 - \mathbf{s}_3\|^2 = a_3^2$$

KINEMATICS OF PARALLEL MANIPULATORS

After expansion, equations take the forms:

$$D_1 c\phi_1 + D_2 c\phi_2 + D_3 c\phi_1 c\phi_2 + D_4 s\phi_1 s\phi_2 + D_5 = 0 \qquad (1)$$

$$E_1 c\phi_2 + E_2 c\phi_3 + E_3 c\phi_2 c\phi_3 + E_4 s\phi_2 s\phi_3 + E_5 \qquad (2)$$

$$F_1 c\phi_1 + F_2 c\phi_3 + F_3 c\phi_1 c\phi_3 + F_4 s\phi_1 s\phi_3 + F_5 = 0 \qquad (3)$$

where c(·) and s(·) stand for cos(·) and sin(·), respectively, while coefficients $\{D_i, E_i, F_i\}_1^5$ functions of the data only and bear the forms shown below:

$$D_1 = 2l_1(\mathbf{o}_2 - \mathbf{o}_1)^{\mathrm{T}}\mathbf{E}\mathbf{u}_1$$

$$D_2 = 2l_2(\mathbf{o}_2 - \mathbf{o}_1)^{\mathrm{T}}\mathbf{E}\mathbf{u}_2$$

$$D_3 = -2l_1 l_2 \mathbf{u}_2^{\mathrm{T}}\mathbf{u}_1$$

KINEMATICS OF PARALLEL MANIPULATORS

$$D_4 = -2l_1 l_2$$

$$D_5 = \|\mathbf{o_2}\|^2 + \|\mathbf{o_1}\|^2 - 2\mathbf{o_1^T o_2} + l_1^2 + l_2^2 - a_1^2$$

$$E_1 = 2l_2(\mathbf{o_3} - \mathbf{o_2})^T \mathbf{Eu_2}$$

$$E_2 = -2l_3(\mathbf{o_3} - \mathbf{o_2})^T \mathbf{Eu_3}$$

$$E_3 = -2l_2 l_3 \mathbf{u_3^T u_2}$$

$$E_4 = -2l_2 l_3$$

$$E_5 = \|\mathbf{o_3}\|^2 + \|\mathbf{o_1}\|^2 - 2\mathbf{o_3^T o_2} + l_2^2 + l_3^2 - a_2^2$$

$$F_1 = 2l_1(\mathbf{o_1} - \mathbf{o_3})^T \mathbf{Eu_1}$$

$$\mathrm{F}_2 = -2l_3(\mathbf{o_1} - \mathbf{o_3})^T \mathbf{Eu_3}$$

$$F_3 = -2l_1 l_3 \mathbf{u_3^T u_1}$$

$$F_4 = -2l_1 l_3$$

$$F_5 = \|\mathbf{o_3}\|^2 + \|\mathbf{o_1}\|^2 - 2\mathbf{o_3^T o_1} + l_1^2 + l_3^2 - a_3^2$$

KINEMATICS OF PARALLEL MANIPULATORS

Our next step is to reduce the foregoing system of three equations in three unknowns to two equations in two unknowns, and hence, obtain two contours in the plane of two of the three unknowns, the desired solutions being determined as the intersections of the two contours.

Since the first equation is already free of Φ_3, all we have to do is eliminate it from the other equations.

The resulting equations take on the forms:

$$k_1\tau_3^2 + k_2\tau_3 + k_3 = 0$$

$$m_1\tau_3^2 + m_2\tau_3 + m_3 = 0$$

KINEMATICS OF PARALLEL MANIPULATORS

Where k_1, k_2, and k_3 are linear combinations of $s\phi_2$, $c\phi_2$, and 1. Likewise, m_1, m_2, and m_3 are linear combinations of $s\phi_1$, $c\phi_1$ and 1, namely,

$$k_1 = E_1 c\phi_2 - E_2 - E_3 c\phi_2 + E_5$$

$$k_2 = 2E_4 s\phi_2$$

$$k_3 = E_1 c\phi_2 + E_2 + E_3 c\phi_2 + E_5$$

$$m_1 = F_1 c\phi_1 - F_2 - F_3 c\phi_1 + F_5$$

$$m_2 = 2F_4 s\phi_1$$

$$m_3 = F_1 c\phi_1 + F_2 + F_3 c\phi_1 + F_5$$

KINEMATICS OF PARALLEL MANIPULATORS

We proceed now by multiplying each of the above equations by τ_3 to obtain two more equations, namely.

$$k_1\tau_3^2 + k_2\tau_3^2 + k_3\tau_3 = 0$$
$$m_1\tau_3^2 + m_2\tau_3^2 + m_3\tau_3 = 0$$

Further, we write equations in homogeneous form as:

$$\mathbf{\Phi}\boldsymbol{\tau}_3 = \mathbf{0}$$

with the 4 x 4 matrix $\mathbf{\Phi}$ and the 4-dimensional vector $\boldsymbol{\tau}_3$ defined as:

$$\mathbf{\Phi} = \begin{bmatrix} k_1 & k_2 & k_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ 0 & k_1 & k_2 & k_3 \\ 0 & m_1 & m_2 & m_3 \end{bmatrix} \qquad \boldsymbol{\tau}_3 = \begin{bmatrix} \tau_3^3 \\ \tau_3^2 \\ \tau_3 \\ 1 \end{bmatrix}$$

KINEMATICS OF PARALLEL MANIPULATORS

Moreover, in view of the form of vector $\boldsymbol{\tau}_3$, we are interested only in nontrivial solutions, which exist only if det($\boldsymbol{\Phi}$) vanishes. We thus have the condition

$$\det(\boldsymbol{\Phi}) = 0 \tag{4}$$

Equations (1) and (4) form a system of two equations in two unknowns ϕ_1 and ϕ_2. Moreover for ϕ_3:

$$\begin{bmatrix} E_4 s\phi_2 & E_2 + E_3 c\phi_2 \\ F_4 s\phi_1 & F_2 + F_3 c\phi_1 \end{bmatrix} \begin{bmatrix} s\phi_3 \\ c\phi_3 \end{bmatrix} = \begin{bmatrix} -E_1 c\phi_2 - E_5 \\ -F_1 c\phi_1 - F_5 \end{bmatrix}$$

Knowing the angles ϕ_1 , ϕ_2 and ϕ_3 allows us to determine the position vectors of the three vertices of the mobile plate $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$. Since three points define a plane, the pose of the EE is uniquely determined by the positions of its three vertices

CONTOUR-INTERSECTION APPROACH

We derive the direct kinematics of a manipulator analyzed by Nanua et al. (1990). This is a platform manipulator whose base plate has six vertices with coordinates expressed with respect to the fixed reference frame $\mathcal{F}_0$ as given below, with all data given in meters:

$A_1 = (-2.9,\ -0.9)$, $B_1 = (-1.2,\ 3.0)$

$A_2 = (\ 2.5,\ 4.1)$, $B_2 = (\ 3.2,\ 1.0)$

$A_3 = (\ 1.3,\ -2.3)$, $B_3 = (-1.2, -3.7)$

The dimensions of the movable triangular plate are, in turn,

$a_1 = 2.0$, $a_2 = 2.0$, $a_3 = 3.0$

CONTOUR-INTERSECTION APPROACH

Determie all possible poses of the moving plate for the six leg-lenghts given as:

$q_{1a} = 5.0$, $q_{2a} = 5.5$, $q_{3a} = 5.7$

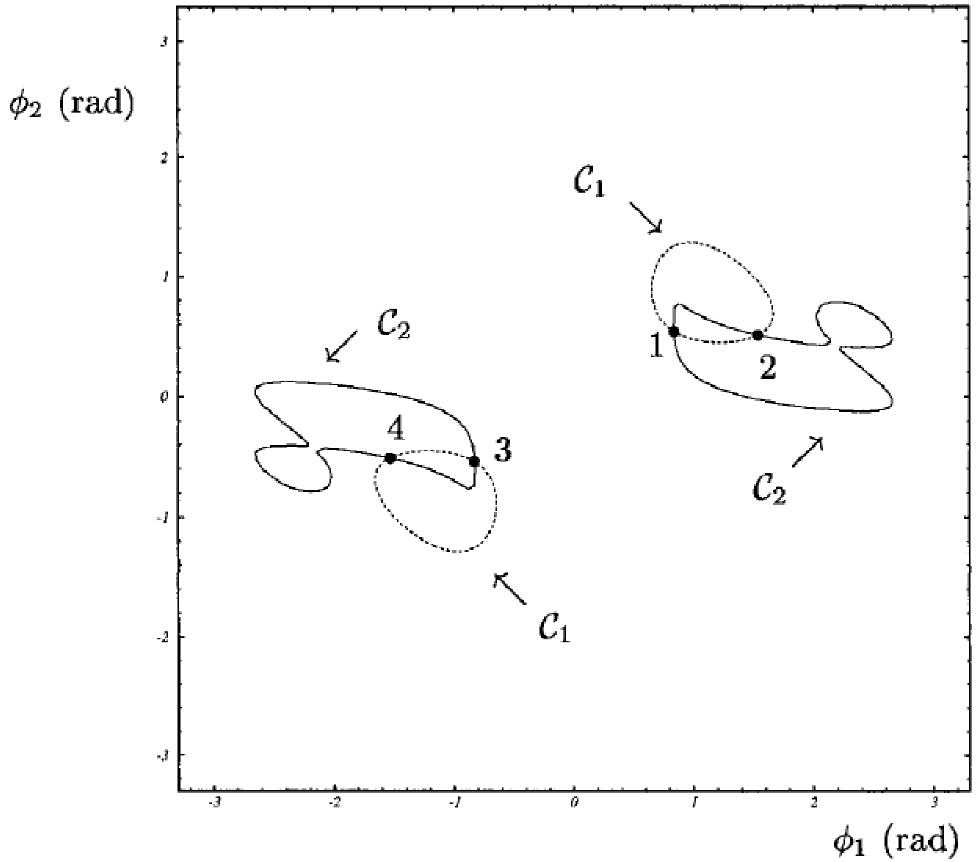
$q_{1b} = 4.5$, $q_{2b} = 5.0$, $q_{3b} = 5.5$

SOLUTION: After substitution of the given numerical values:

$$61.848 - 36.9561c_1 - 47.2376c_2 + 33.603c_1c_2 - 41.6822s_1s_2 = 0$$

$$\begin{aligned}
&-28.5721 + 48.6806c_1 - 20.7097c_1^2 + 68.7942c_2 - 100.811c_1c_2 \\
&+ 35.9634c_1^2c_2 - 41.4096c_2^2 + 50.8539c_1c_2^2 - 15.613c_1^2c_2^2 \\
&- 52.9786s_1^2 + 67.6522c_2s_1^2 - 13.2765c_2^2s_1^2 + 74.1623s_1s_2 \\
&- 25.6617c_1s_1s_2 - 67.953c_2s_1s_2 + 33.9241c_1c_2s_1s_2 - 13.202s_2^2 \\
&- 3.75189c_1s_2^2 + 6.13542c_1^2s_2^2 = 0
\end{aligned}$$

CONTOUR-INTERSECTION APPROACH



THE UNIVARIATE POLYNOMIAL APPROACH

Reduce the two equations above, to a single monovariate polynomial equation.

SOLUTION:

We obtain two polynomial equations in τ_1, namely,

$$G_1\tau_1^4 + G_2\tau_1^3 + G_3\tau_1^2 + G_4\tau_1 + G_5 = 0$$

$$H_1\tau_1^2 + H_2\tau_1 + H_3 = 0$$

No .	ϕ_1 (rad)	ϕ_2 (rad)	ϕ_3 (rad)
1	0.8335	0.5399	0.8528
2	1.5344	0.5107	0.2712
3	-0.8335	-0.5399	-0.8528
4	-0.5107	-0.5107	-0.2712

THE UNIVARIATE POLYNOMIAL APPROACH

Where:

$$G_1 = K_1\tau_2^4 + K_2\tau_2^2 + K_3$$
$$G_2 = K_4\tau_2^3 + K_5\tau_2$$
$$G_3 = K_6\tau_2^4 + K_7\tau_2^2 + K_8$$
$$G_4 = K_9\tau_2^3 + K_{10}\tau_2$$
$$G_5 = K_{11}\tau_2^4 + K_{12}\tau_2^2 + K_{13}$$
$$H_1 = L_1\tau_2^2 + L_2$$
$$H_2 = L_3\tau_2$$
$$H_3 = L_4\tau_2^2 + L_5$$

THE UNIVARIATE POLYNOMIAL APPROACH

Multiplying the first equation by H_1 and the second by $G_1\tau_1$ and subtract the two equations thus resulting, which leads to a cubic equation in τ_1 , namely,

$$(G_2H_1 - G_2H_2)\tau_1^3 + (G_3H_1 - G_1H_3)\tau_1^2 + G_4H_1\tau_1 + G_5H_1 = 0$$

Likewise multiplying the first by $H_1\ \tau_1$ and the second one by $G_1\ \tau_1{}^3 + G_2\ \tau_1{}^2$ and the equations thus resulting are subtracted from each other, one more cubic equation in τ_1 is obtaines :

$$(G_1H_3 - G_3H_1)\tau_1^3 + (G_4H_1 + G_3H_2 - G_2H_3)\tau_1^2$$
$$+ (G_5H_1 + G_4H_2)\tau_1 + G_5H_2 = 0$$

THE UNIVARIATE POLYNOMIAL APPROACH

Moreover, multiplying the second equation by $\tau_{1,}$ a third cubic equation in τ_1 can be derived:

$$H_1\tau_1^3 + H_2\tau_1^2 + H_3\tau_1 = 0$$

Now the foregoing equations constitute a homogeneous linea system of four equations in the firt four powers of τ_1 which can be cast in the form:

$$\mathbf{H}\boldsymbol{\tau}_1 = \mathbf{0}$$

Where $\boldsymbol{\tau}_1 = [\tau_1^3 \quad \tau_1^2 \quad \tau_1 \quad 1]$

THE UNIVARIATE POLYNOMIAL APPROACH

$$\mathbf{H} = \begin{bmatrix} G_2H_1 - G_2H_2 & G_3H_1 - G_1H_3 & G_4H_1 & G_5H_1 \\ G_3H_1 - G_1H_3 & G_3H_2 - G_2H_3 + G_4H_1 & G_4H_2 + G_5H_1 & G_5H_2 \\ H_1 & H_2 & H_3 & 0 \\ 0 & H_1 & H_2 & H_3 \end{bmatrix}$$

To admit a nontrivial solution, the determinant of its coefficient matrix must vanish:

$$\det(\mathbf{H}) = 0$$

and this can be expanded in the form:

$$\det(\mathbf{H}) = H_1\Delta_1 + H_2\Delta_2 + H_3\Delta_3$$

THE UNIVARIATE POLYNOMIAL APPROACH

Where:

$$\Delta_2 = det\begin{bmatrix} G_2H_1 - G_2H_2 & G_4H_1 & G_5H_1 \\ G_3H_1 - G_1H_3 & G_4H_2 + G_5H_1 & G_5H_2 \\ H_1 & H_2 & 0 \end{bmatrix}$$

$$\Delta_2 = det\begin{bmatrix} G_2H_1 - G_2H_2 & G_3H_1 - G_1H_3 & G_5H_1 \\ G_3H_1 - G_1H_3 & G_3H_2 - G_2H_3 + G_4H_1 & G_5H_2 \\ H_1 & H_2 & 0 \end{bmatrix}$$

$$\Delta_2 = det\begin{bmatrix} G_2H_1 - G_1H_2 & G_3H_1 - G_1H_3 & G_4H_1 \\ G_3H_1 - G_1H_3 & G_3H_2 - G_2H_3 + G_4H_1 & G_4H_2 + G_5H_1 \\ H_1 & H_2 & H_3 \end{bmatrix}$$

VELOCITY ANALYSES OF PARALLEL MANIPULATORS

The inverse velocity analysis of this manipulator consists in determining the six rates of the active joints, $\{\dot{b_k}\}_1^6$ given the twist **t** of the moving platform. The velocity analysis of a typical leg leads to a relation of the form of equation namely,

$$\mathbf{J}_J \dot{\boldsymbol{\theta}}_J = \mathbf{t}_J \qquad\qquad J = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{IV}$$

Where $\mathbf{J}_\mathrm{J}$ is the Jacobian of the Jth leg, is the 6-d joint-rate vector $\dot{\boldsymbol{\theta}}_J$ of the same leg and $\mathbf{t}_\mathrm{j}$ is the twist moving platform $\mathcal{M}$ with its operation point defined as the point C_J

VELOCITY ANALYSES OF PARALLEL MANIPULATORS

We thus have:

$$\mathbf{J_J} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{0} & \mathbf{e_4} & \mathbf{e_5} & \mathbf{e_6} \\ b_{34}\mathbf{e_1} \times \mathbf{e_3} & b_{34}\mathbf{e_2} \times \mathbf{e_3} & \mathbf{e_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{t_J} = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{c}}_\mathbf{J} \end{bmatrix}, \qquad b_{34} = b_3 + b_4$$

it is apparent that

$$\dot{\mathbf{c}}_\mathbf{J} = \dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{r_J}$$

with $\mathbf{r}_J$ defined as the vector directed from C_J to the operation point P of the moving platform.

VELOCITY ANALYSES OF PARALLEL MANIPULATORS

Multiplying the left side of the equation by $\mathbf{I}_J{}^{T}$

Where: $\mathbf{I}_J^T = [\,\mathbf{0}^T \quad \mathbf{e}_3^T\,]$

We obtain:

$$\mathbf{I}_J^T\,\mathbf{J}_J\dot{\boldsymbol{\theta}}_J = \dot{b}_J$$

On the other hand we have:

$$\mathbf{I}_J^T\mathbf{t}_J = \mathbf{e}_J^T\,\dot{\mathbf{c}}_J$$

Upon equating the right-hand sides of the foregoing equations we have: $\dot{\boldsymbol{b}}_J = \mathbf{e}_J^T\,\dot{\mathbf{c}}_J$

VELOCITY ANALYSES OF PARALLEL MANIPULATORS

We obtain the relations between the actuated joint rates and the twist of the moving platform, namely

$$\dot{b}_j = \begin{bmatrix}(\mathbf{e}_\mathrm{J} \times \mathbf{t}_\mathrm{j})^T & \mathbf{e}_\mathrm{J}^\mathrm{T}\end{bmatrix}\begin{bmatrix}\omega \\ \dot{\mathbf{c}}\end{bmatrix} = \mathbf{k}_j^T \mathbf{t}$$

We have the desired relation between the vector of actuated joint rates and the twist of the moving platform, namely.

$$\dot{\mathbf{b}} = \mathbf{K}\mathbf{t}$$

With the 6-d vectors **b** and **t** defined as the vector of joint variables and the twist of the platform at the operation point, respectively. Moreover, the 6 x 6 matrix **K** is the Jacobian of the manipulator at hand.

VELOCITY ANALYSES OF PARALLEL MANIPULATORS

These quantities are displayed below:

$$\mathbf{b} = \begin{bmatrix} b_I \\ b_{II} \\ \vdots \\ b_{IV} \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} \omega \\ \dot{\mathbf{p}} \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} (\mathbf{e}_{\mathrm{I}} \times \mathbf{r}_{\mathrm{I}})^{\mathrm{T}} & \mathbf{e}_{\mathrm{I}}^{\mathrm{T}} \\ (\mathbf{e}_{\mathrm{II}} \times \mathbf{r}_{\mathrm{II}})^{\mathrm{T}} & \mathbf{e}_{\mathrm{II}}^{\mathrm{T}} \\ \vdots & \vdots \\ (\mathbf{e}_{\mathrm{VI}} \times \mathbf{r}_{\mathrm{VI}})^{\mathrm{T}} & \mathbf{e}_{\mathrm{VI}}^{\mathrm{T}} \end{bmatrix}$$

ACCELLERATION ANALYSES OF PARALLEL MANIPULATORS

The acceleration analysis of the same leg is straightforward. Indeed, upon differentiation of both sides of equation with respect to time, one obtains

$$\ddot{\mathbf{b}} = \mathbf{K}\dot{\mathbf{t}} + \dot{\mathbf{K}}\mathbf{t}$$

Where $\dot{\mathbf{K}}$ takes the form:

$$\dot{\mathbf{K}} = \begin{bmatrix} \dot{\mathbf{u}}_{I}^{T} & \dot{\mathbf{e}}_{I}^{T} \\ \dot{\mathbf{u}}_{II}^{T} & \dot{\mathbf{e}}_{II}^{T} \\ \vdots & \vdots \\ \dot{\mathbf{u}}_{VI}^{T} & \dot{\mathbf{e}}_{VI}^{T} \end{bmatrix}$$

ACCELLERATION ANALYSES OF PARALLEL MANIPULATORS

$\mathbf{u}_J$ is defined as: $\mathbf{u}_\mathrm{J} = \mathbf{e}_\mathrm{J} \times \mathbf{r}_\mathrm{J}$

Therefore: $\dot{\mathbf{u}}_\mathrm{J} = \dot{\mathbf{e}}_\mathrm{J} \times \mathbf{r}_\mathrm{J} + \mathbf{e}_\mathrm{J} \times \dot{\mathbf{r}}_\mathrm{J}$

Now since the vector $\mathbf{r}_j$ are fixed to the moving platform, their time-derivatives are simply given by:

$$\dot{\mathbf{r}}_\mathrm{J} = \boldsymbol{\omega} \times \mathbf{r}_\mathrm{J}$$

On the other hand, vector ej is directed along the log axis, and so, its time-derivative is given by:

$$\dot{\boldsymbol{\omega}}_\mathrm{J} = \boldsymbol{\omega}_\mathrm{J} \times \mathbf{e}_\mathrm{J}$$

with ω_J defined as the angular velocity of the third leg link:

$$\boldsymbol{\omega}_J = \left(\dot{\theta}_1 \boldsymbol{e}_1 + \dot{\theta}_2 \boldsymbol{e}_2\right)_J$$

ACCELLERATION ANALYSES OF PARALLEL MANIPULATORS

In order to simplify the notation, we start by defining

$$\mathbf{f}_{\mathrm{J}} = (\mathbf{e}_1)_{\mathrm{J}} \qquad \mathbf{g}_{\mathrm{J}} = (\mathbf{e}_2)_{\mathrm{J}}$$

Now we write the second vector of the equation $\mathbf{J}_J \dot{\boldsymbol{\theta}}_J = \mathbf{t}_J$ using the foregoing definitions, which yields:

$$(\dot{\theta_1})_{\mathrm{J}} \mathbf{f}_{\mathrm{J}} \times \left(\mathrm{b}_{\mathrm{J}} + \mathrm{b}_4\right)\mathbf{e}_{\mathrm{j}} + (\dot{\theta_2})_{\mathrm{J}} \mathbf{g}_{\mathrm{J}} \times \left(\mathrm{b}_{\mathrm{J}} + \mathrm{b}_4\right)\mathbf{e}_{\mathrm{j}} + \dot{\mathrm{b}}_{\mathrm{J}} \mathbf{e}_{\mathrm{j}} = \dot{\mathbf{c}}_{\mathrm{J}}$$

where b_4 is the same for all legs, since all have identical architectures. Now we can eliminate $(\dot{\theta_2})_{\mathrm{J}}$ from the foregoing equation by dot-multiplying its two sides by g j , thereby producing

ACCELLERATION ANALYSES OF PARALLEL MANIPULATORS

$$(\dot{\theta_1})_J \mathbf{g_J} \times \mathbf{f_J} \cdot (b_J + b_4)\mathbf{e_j} + \mathbf{g_J^T}(\mathbf{e_j}\mathbf{e_J^T})\dot{\mathbf{c}}_J = \mathbf{g}^T\dot{\mathbf{c}}_J$$

Now it is a simple matter to solve for $(\dot{\theta_1})_J$ from the above equation, which yields:

$$(\dot{\theta_1})_J = \frac{g_J^T(1 - e_J e_J^T)\dot{c}_J}{\Delta_J}$$

Where: $\Delta_J = (b_J + b_4)\mathbf{e_j} \times \mathbf{f_J} \cdot \mathbf{g_J}$

ACCELLERATION ANALYSES OF PARALLEL MANIPULATORS

Moreover, we can obtain the above expression for $(\dot{\theta_1})_J$ in terms of the platform twist by recalling equation which is reproduced below in a more suitable form for quick reference:

$$\dot{\mathbf{c}}_J = \mathbf{C}_J \mathbf{t}$$

where **t** is the twist of the platform, the 3 x 6 matrix $\mathbf{C}_j$ being defined as:

$$\mathbf{C}_J = [\mathbf{R}_j \quad \mathbf{1}]$$

In which R_J is the cross-product matrix of r_J

ACCELLERATION ANALYSES OF PARALLEL MANIPULATORS

Therefore the expressione sought for $(\dot{\theta_1})_\mathrm{J}$ takes the form:

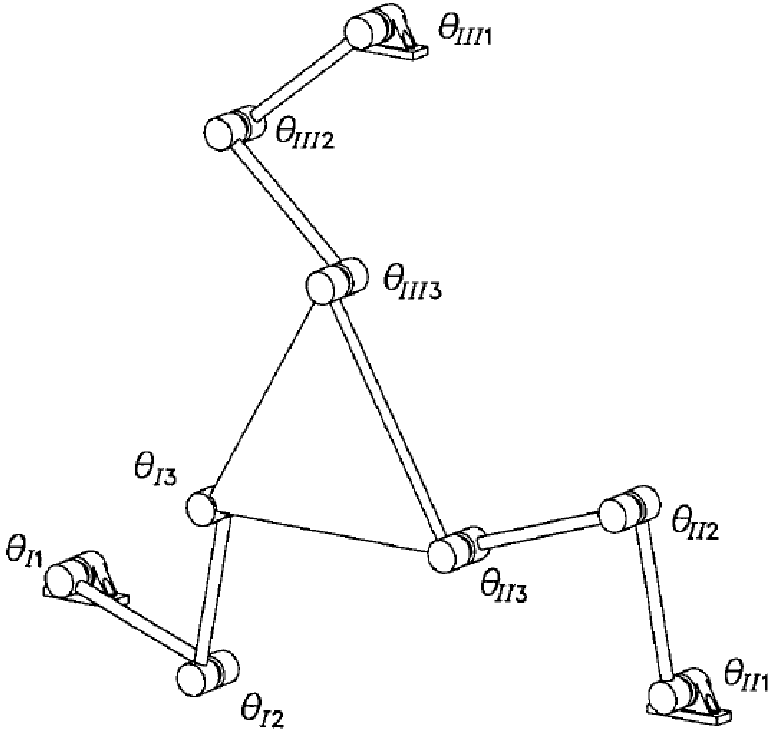
$$(\dot{\theta_1})_\mathrm{J} = \frac{1}{\Delta_J}\mathbf{g}_J^T(1 - \mathbf{e}_\mathrm{J}\mathbf{e}_\mathrm{J}^\mathrm{T})\mathbf{C}_\mathrm{J}\mathbf{t}$$

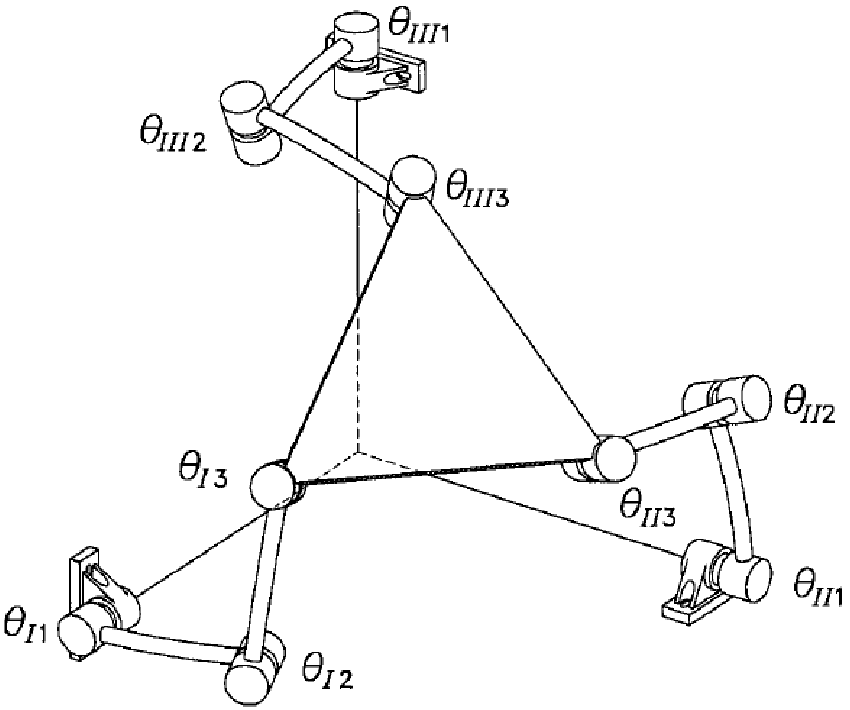
A similar procedure can be followed to find $(\dot{\theta_2})_\mathrm{J}$ the final result being

$$(\dot{\theta_2})_\mathrm{J} = \frac{1}{\Delta_J}\boldsymbol{f}_J^T(1 - \mathbf{e}_\mathrm{J}\mathbf{e}_\mathrm{J}^\mathrm{T})\mathbf{C}_\mathrm{J}\mathbf{t}$$

PLANAR AND SPHERICAL MANIPULATORS

The velocity analysis of the planar and spherical parallel manipulators of figures are outlined below





PLANAR MANIPULATORS

$$\mathbf{J}_J \dot{\boldsymbol{\theta}}_J = \mathbf{t}_J \qquad \text{J= I,II, III}$$

Where $\mathbf{J}_J$ is the Jacobian matrix of this leg while $\dot{\boldsymbol{\theta}}_J$ is the 3-D vectore of joint rates of this leg:

$$J_J = \begin{bmatrix} 1 & 1 & 1 \\ Er_{J1} & Er_{J2} & Er_{J3} \end{bmatrix} \qquad \theta_J^{\cdot} = \begin{bmatrix} \theta_{J1}^{\cdot} \\ \theta_{J2}^{\cdot} \\ \theta_{J3}^{\cdot} \end{bmatrix}$$

We multiply both sides of the said equation by a 3-D vector $\mathbf{n}_j$ perpendicular to the second and the third columns of J_J.

PLANAR MANIPULATORS

This vector can be most easily determined as the cross product of those two columns, namely, as

$$\mathbf{n} = \mathbf{j}_{J2} \times \mathbf{j}_{J3} = \begin{bmatrix} -\mathbf{r}_{J2}^{T}\mathbf{E}\mathbf{r}_{J3} \\ \mathbf{r}_{J2} - \mathbf{r}_{J3} \end{bmatrix}$$

Upon multiplication of both side of the equation by $\mathbf{n}_j^{T}$, we obtain:

$$\left[-\mathbf{r}_{J2}\mathbf{E}\mathbf{r}_{j3} + \left(\mathbf{r}_{J2} - \mathbf{r}_{J3}\right)^{T}\mathbf{E}\mathbf{r}_{J1}\right]\dot{\theta}_{J1} = -\left(\mathbf{r}_{J2}\mathbf{E}\mathbf{r}_{J3}\right)\omega + \left(\mathbf{r}_{J2} - \mathbf{r}_{J3}\right)^{T}\dot{\mathbf{c}}$$

And hence we can solve directly reriving:

$$\dot{\theta}_{J1} = \frac{-\left(\mathbf{r}_{J2}\mathbf{E}\mathbf{r}_{J3}\right)\omega + \left(\mathbf{r}_{J2} - \mathbf{r}_{J3}\right)^{T}\dot{\mathbf{c}}}{\left[-\mathbf{r}_{J2}\mathbf{E}\mathbf{r}_{j3} + \left(\mathbf{r}_{J2} - \mathbf{r}_{J3}\right)^{T}\mathbf{E}\mathbf{r}_{J1}\right]}$$

PLANAR MANIPULATORS

The foregoing equation can be written in the form:

$$j_J \dot{\theta}_{J1} = \mathbf{k}_\mathrm{J}^\mathrm{T} \mathbf{t} \qquad \mathrm{J=I,II,\ III}$$

With j_J and $\mathbf{k}_\mathrm{J}$ defined:

$$j_J = \left(\mathbf{r}_{\mathrm{J}2} - \mathbf{r}_{\mathrm{J}3}\right)^\mathrm{T} \mathbf{E}\mathbf{r}_{\mathrm{J}1} - \mathbf{r}_{\mathrm{J}2}\mathbf{E}\mathbf{r}_{\mathrm{j}3}$$

$$\mathbf{k}_\mathrm{J} = \left[\mathbf{r}_{\mathrm{J}2}^\mathrm{T}\mathbf{E}\mathbf{r}_{\mathrm{J}3} \quad \left(\mathbf{r}_{\mathrm{J}2} - \mathbf{r}_{\mathrm{J}3}\right)^\mathrm{T}\right]^\mathrm{T}$$

If we further define:

$$\dot{\boldsymbol{\theta}} = [\dot{\theta}_{I1} \quad \dot{\theta}_{II2} \quad \dot{\theta}_{III3}]$$

And assemble all three foregoing joint-rate-twist relation we obtain: $\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{K}\mathbf{t}$

PLANAR MANIPULATORS

$$\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{K}\mathbf{t}$$

Where **J** and **K** are the two manipulator Jacobians defined as:

$$\mathbf{J} = \mathrm{diag}\left(j_{\mathrm{I}}, j_{\mathrm{II}}, j_{\mathrm{III}}\right), \qquad \mathbf{K} = \begin{bmatrix} \mathbf{k}_{\mathrm{I}}^{\mathrm{T}} \\ \mathbf{k}_{\mathrm{II}}^{\mathrm{T}} \\ \mathbf{k}_{\mathrm{III}}^{\mathrm{T}} \end{bmatrix}$$

Expressions for the joint accelerations can be readily derived by differentiation of the foregoing expressions with respect to time.

SPHERICAL MANIPULATOR

The velocity analysis of the spherical parallel manipulator can be accomplished similarly. Thus, the velocity relations of the Jth leg take on the form:

$$\mathbf{J}_{\mathrm{J}}\dot{\boldsymbol{\theta}}_{\mathrm{J}} = \boldsymbol{\omega} \qquad\qquad J=I,II,III$$

Where the Jacobian of the Jth leg $\mathbf{J}_{\mathrm{J}}$ is defined as:

$$J_J = [\mathbf{e}_{J1} \quad \mathbf{e}_{J2} \quad \mathbf{e}_{J3}]$$

While the joint-rate vector of the Jth $\dot{\boldsymbol{\theta}}_{\mathrm{J}}$leg is defined exactly as in the planar case analyzed above.

An obvious definition of this vector is, then:

$$\mathbf{n}_{\mathrm{J}} = \mathbf{e}_{\mathrm{J}2} \times \mathbf{e}_{\mathrm{J}3}$$

SPHERICAL MANIPULATOR

The desired joint-rate relation is thus readily derived as:

$$j_J \dot{\theta_{J1}} = \mathbf{k}_\mathrm{J}^\mathrm{T} \boldsymbol{\omega}$$

where j_J and $\mathbf{k}_J$ are now defined as:

$$j_J = \mathbf{e}_{J1} \times \mathbf{e}_{J2} \cdot \mathbf{e}_{J3}$$

$$\mathbf{k}_J = \mathbf{e}_{J2} \times \mathbf{e}_{J3}$$

The accelerations of the actuated joints can be derived, again, by differentiation of the foregoing expressions. We can then say that in general, parallel manipulators, as opposed to serial ones, have two Jacobian matrices.